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Determination of convex bodies by certain sets of sectional volumes

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Abstract

Suppose that K and L are convex bodies in $\mathbb{R}^n$ with $L \subset \operatorname{int} K$. For each hyperplane $H \subset \mathbb{R}^n$ which supports L, define $f_{K,L}(H) = |K \cap H|$, where $|\cdot|$ denotes the $n-1$-dimensional Lebesgue measure. We conjecture that K is determined uniquely by $f_{K,L}$. A number of partial results are presented, and some obvious alternative formulations are shown to be trivial. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

At the 1961 AMS Symposium on Convexity, Hammer [7] proposed the X-ray problem which now bears his name. Suppose that $P \subset \mathbb{R}^n$ and let S denote the set of lines which pass through at least one point of P. If $K \subset \mathbb{R}^n$ is a convex body, define the X-ray picture $X_{K,S}$ of K with respect to S by

$$X_{K,S} : S \to \mathbb{R}$$

$$X_{K,S}(l) = |K \cap l|,$$

where $|\cdot|$ denotes Euclidean length. Hammer asked how large P must be in order to guarantee that every convex body K can be determined uniquely by its X-ray picture with respect to S. Formally, what is the minimal cardinality of P such that whenever $K, M \subset \mathbb{R}^n$ are convex bodies, $X_{K,S}$ and $X_{M,S}$ are identical only if $K = M$?

In the same spirit as this first question, Hammer also posed the problem in the case that the points of P are transported to infinity. Let D be a set of directions in $\mathbb{R}^n$ and S the set of lines parallel to one or more members of D. What is the minimum cardinality of D such that whenever $K, M \subset \mathbb{R}^n$ are convex bodies, $X_{K,S}$ and $X_{M,S}$ are identical only if $K = M$?

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During the last three or four decades, significant progress has been made in geometric tomography. In particular, many articles relating to Hammer's X-ray problem have been published. In [6] Gardner and McMullen study the projective version and establish conditions under which $|D| = 4$ suffices. Falconer [3,4] gives a partial solution to the point X-ray problem, showing that, under certain conditions, $|P| = 2$ suffices. In these papers Falconer also proves generalisations involving higher dimensional sections. More recently Volčič [14] provided three- and four-point solutions. The results presented in [14] are particularly interesting in that they apply equally to both forms of Hammer's problem.

One obvious generalisation of the point X-ray problem involves sections instead of chords. Let $1 < k < n$, replace S by the set of k-dimensional affine subspaces of $\mathbb{R}^n$ containing a point of P, and understand $|\cdot|$ to mean the k-dimensional Lebesgue measure. Most often, formulations such as this can be reduced relatively quickly to problems involving chords through the use of spherical harmonics or other devices. Gardner [5] gives an excellent survey of the fundamental techniques in this area.

A related class of tomographic results involve characterisation of the ellipsoid. A significant representative of results in this area, The False Centre Theorem, was given by Aitchison et al. in [1]. A point c is said to be a false centre of the convex body $K \subset \mathbb{R}^n$ ($n \geqslant 3$), if every one-codimensional section of K through c is centrally symmetric, but K is not centrally symmetric about c. The False Centre Theorem states that a convex body K with a false centre $c \in \operatorname{int} K$ is an ellipsoid. Larman [9] later extended this result to cover the case in which $c \notin \operatorname{int} K$.

The problems mentioned above concern sections taken through a point, or set of points. A much older result given by Olovjanischnikoff [12] in 1941 highlights the potential for a rather different formulation. Let $K \subset \mathbb{R}^n$ be a convex body, and $0 < \varepsilon < 1$. A one-codimensional section H of K is said to be an ε-section of K if H divides the volume of K in the ratio $\varepsilon : 1$. Olovjanischnikoff showed that if every ε-section of K is centrally symmetric, then K is an ellipsoid. More recently, Meyer and Reisner [11] showed that the sections considered by Olovjanischnikoff have an attractive property. When K is centrally symmetric, the ε-sections of K are exactly those cut by the supporting hyperplanes of a uniquely defined convex body K_ε.

These two facts led us to wonder whether Olovjanischnikoff's result could be modified in the following way:

Conjecture 1. Suppose that $K, L \subset \mathbb{R}^n$ are convex bodies with $n \geqslant 3$ and $L \subset \operatorname{int} K$. Suppose further that whenever H is a hyperplane supporting L the section $H \cap K$ of K is centrally symmetric. Then K is an ellipsoid.

This conjecture remains unproved, although a much restricted version in three dimensions is proved by Barker in [2]. During exploration of this conjecture it soon became apparent that a similar modification of Hammer's X-ray problem might prove useful. Moreover, it is probably more interesting and significant in itself:

Conjecture 2. Suppose that $K_1, K_2, L \subset \mathbb{R}^n$ are convex bodies with $L \subset \operatorname{int} K_1 \cap \operatorname{int} K_2$; further assume that whenever $H \subset \mathbb{R}^n$ is a hyperplane supporting L, the $(n-1)$-volumes $|K_1 \cap H|$ and $|K_2 \cap H|$ are equal. Then $K_1 = K_2$.

In the present work, a number of partial results relating to this conjecture are given. We have not been able to prove the full conjecture, but several weaker versions of the conjecture are shown to be trivial.

One result not presented in this paper is worth mentioning in passing. If L above is replaced by a pair of suitably constrained convex bodies L_1, L_2, the two-dimensional version of Conjecture 2 can be shown to be locally equivalent to the two point X-ray problem considered by Falconer. The constraint needed guarantees that the topology exposed matches that of the two-point X-ray problem: whenever l is a supporting line of both L_1 and L_2, the sets $l \cap L_1$ and $l \cap L_2$ do not intersect. For this reason, a proof of this result is not presented here. Interested readers may consult [2] for details.

In what follows we treat only the special case in which L is a Euclidean ball. This makes calculations much easier but does not impact on the topological complexity of the problem.

2. Definitions and notation

The following notation is used throughout. Most of it is standard.

Let $X \subset \mathbb{R}^n$. The closure and affine hull of X are denoted by $\operatorname{cl} X$ and $\operatorname{aff} X$, respectively. The interior of X is written $\operatorname{int} X$ and the boundary of X is defined by $\partial X = \operatorname{cl} X \backslash \operatorname{int} X$.

If $X \subset \mathbb{R}^n$ is Lebesgue measurable and $\dim \operatorname{aff}(X) = j$ then $|X|$ is the j-dimensional Lebesgue measure of X. During integration, the j-dimensional Lebesgue measure is written λ_j, and the $(j-1)$-dimensional spherical surface measure as ω_{j-1}.

The Euclidean norm is written $\|\cdot\|$ as usual.

The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}. The Euclidean ball is always written B, its dimension being clear from the context.

By a *convex body* $K \subset \mathbb{R}^n$ we understand a convex subset of $\mathbb{R}^n$ which is bounded and has non-empty interior. Let $K \subset \mathbb{R}^n$ be a convex body. The radial function of K is written ρ_K. We define f_K to be the function which maps affine subspaces of $\mathbb{R}^n$ to the Lebesgue measure of their intersection with K. That is

$$f_K(A) = |K \cap A|.$$

If $L \subset \mathbb{R}^n$ is another convex body (usually contained in the interior of K), then $f_{K,L}$ denotes the restriction of f_K to the set of hyperplanes which support L.

For planar convex bodies $L \subset K \subset \mathbb{R}^2$, the map $T_{K,L}$ is the *chord-map* of K with respect to L. The precise definition of the chord-map is introduced in the section below.

Finally, for the sake of clarity, we remark that by j-plane we mean a j-dimensional affine subspace.

3. The chord-map

In common with other tomographic results, the two-dimensional version of Conjecture 2 appears just as difficult as higher-dimensional versions. Inspired by Falconer's work on point X-ray problems, a number of the planar results that follow will make use of a device known as the *chord-map*. In the present context, if K, L are convex bodies in the plane with $L \subset \operatorname{int} K$, the chord-map $T_{K,L}$ of K with respect to L is defined as follows:

Let $x \in \mathbb{R}^2 \backslash L$ be near to the boundary of K. There are exactly two lines l_1 and l_2 containing x which support L. If l_1 and l_2 meet L at x_1 and x_2, respectively, suppose without loss of generality that the triple (x_1, x_2, x) describes the vertices of a triangle in the positive sense. Now $f = f_{K,L}(l_1)$ is the length of the chord of K cut by l_1. $T_{K,L}(x)$ is defined as the point on l_1 with x_1 in the interior of the line segment joining $T_{K,L}(x)$ to x which satisfies $\|x - T_{K,L}(x)\| = f$. See Fig. 1 for an illustration which should help to clarify this construction.

There are some difficulties with the domain of definition of the chord-map. It will usually suffice to specify the domain of $T_{K,L}$ as a region near to the boundary of K with the precise details apparent from the context.

Note that $T_{K,L}$ is determined uniquely by (L and) the chord-lengths of K supporting L, so in Conjecture 2, $T_{K_1,L}$ and $T_{K_2,L}$ are identical. Also, the boundaries of K_1 and K_2

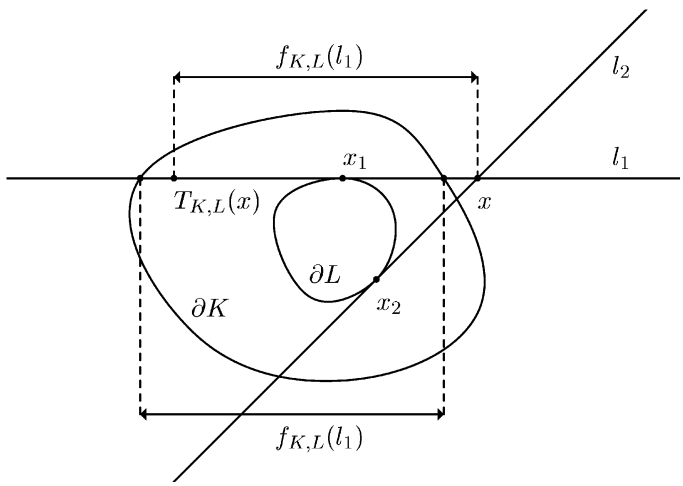

Fig. 1. The chord-map.

are invariant under the action of the chord-map. Hence, it is possible that the conjecture could be rewritten in terms of the possible existence of more than one invariant circle of a given map.

4. Results

First consider Conjecture 2 in the plane, and in particular the simplest case possible. If K is a Euclidean ball containing B, and K and B have the same centre, then the chords of K supporting B have constant length. The first result proves the converse. Notice that this contrasts with the corresponding point X-ray problem; every chord through the centre of a circle has the same length, but there exist many convex bodies with constant chord-length through a single point.

4.1. Determination of the circle

Theorem 1. *Suppose that $K \subset \mathbb{R}^2$ is a convex body containing, in its interior, B. Then if the chords of K cut by supporting lines of B have constant length, K is a unique multiple of B.*

Proof. The proof of this statement is easily obtained by using the properties of the chord-map. After choice of suitable coordinates, the chord map has a particularly simple form. Use coordinates (θ, r) defined by

$$x(\theta, r) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + r \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

and suppose that K has chords of length $c > 0$ supporting B. The chord-map $T_{K,B}$ can now be written as

$$T_{K,B}(\theta, r) = (\theta + 2\tan^{-1}(c - r), c - r).$$

Fig. 2 illustrates the coordinates and chord-map. $T_{K,B}$ maps ∂K to ∂K; hence so does $T^2_{K,B}$:

$$T^2_{K,B}(\theta, r) = (\theta + 2v(r), r),$$

where $v(r) = \tan^{-1} r + \tan^{-1}(c - r)$. Thus $T^2_{K,B}$ has the effect of increasing θ by a quantity dependent only on r, and keeping r constant. It is in effect a cylindrical shear. In the literature, this type of map is sometimes referred to as a twist-map. After $2n$ applications of $T_{K,B}$,

$$T^{2n}_{K,B}(\theta, r) = (\theta + 2nv(r), r).$$

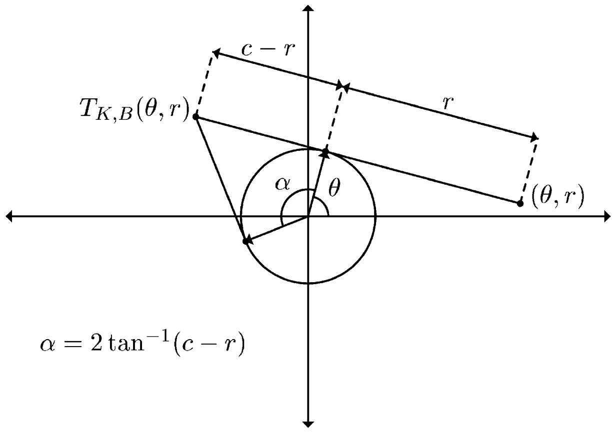

Fig. 2. The chord-map used in Theorem 1.

Now if $(\theta, r) \in \partial K$, the set

$$\{(\theta + 2nv(r), r) \mid n \in \mathbb{N}\}$$

is a subset of ∂K. Suppose now that $v = v(r)$ is an irrational multiple of π. In this case, the set

$$\{\theta + 2nv(\operatorname{mod} 2\pi) \mid n \in \mathbb{N}\}$$

is dense in the interval $[0, 2\pi)$ (see [8, Proposition 1.3.3 for example]). This completes the proof, for unless ∂K is already a circle, there is surely some $(\theta, r) \in \partial K$ with $v(r)/\pi$ irrational. Closure of ∂K now implies that ∂K is a circle with centre 0, and only one such circle has chords of length c supporting the unit ball.

4.2. The general chord-map

Returning to the general two-dimensional problem, consider again the action of the chord-map. Using the coordinates (θ, r) defined in Theorem 1 it is possible to derive a more general form for $T_{K,B}$.

Let K be a convex body in the plane containing the unit ball. Suppose that the boundary of K is given by

$$r(\theta) = \max\{\lambda > 0 \mid x(\theta, \lambda) \in K\}.$$

Let $l(\theta)$ denote the length of the chord of K supporting B at $\xi(\theta) = (\cos\theta, \sin\theta)^T$. Elementary geometry yields the relation

$$l(\theta) = r(\theta) + r(\theta + 2\tan^{-1}(l(\theta) - r(\theta))). \tag{1}$$

Given (θ, r) with r close to $r(\theta)$ set

$$T_{K,B}(\theta, r) = (\Phi(\theta, r), R(\theta, r)), \tag{2}$$

with Φ and R defined by

$$R(\theta,r)=l(\theta)-r,$$

$$\Phi(\theta,r)=\theta+2\tan^{-1}R(\theta,r).$$

Investigation of the properties of $T_{K,B}$, for given K, soon leads one to ask questions concerning the existence of periodic points of ∂K under iterates of $T_{K,B}$. Specifically, for which values of $n\in\mathbb{N}$ is it possible to find $x\in\partial K$ with $T^n_{K,B}(x)=x$? Of course, this situation is equivalent to finding a polygon inscribed in K with edges supporting B. The next section is devoted to providing some answers in this direction.

4.3. Rotation numbers and periodic points

A concept from the study of dynamics of homeomorphisms of the circle, known as the rotation number, will prove useful. Identify the circle S^1 with $\mathbb{R}/\mathbb{Z}$ using the map $\Pi : x\mapsto x(\bmod 1)$. If $f:S^1\to S^1$ is a homeomorphism, and $\tilde f:\mathbb{R}\to\mathbb{R}$ is a homeomorphism with the property

$$\Pi\tilde f(x)=f(\Pi x)$$

for all $x\in\mathbb{R}$, say that $\tilde f$ is a *lift* of f. Given any lift $\tilde f$ of f, define the rotation number $\rho(f)$ of f by

$$\rho(f)=\lim_{n\to\infty}\left(\frac{\tilde f^n(x)-x}{n}\right)(\bmod 1).$$

The quantity $\rho(f)$ can be shown to be independent of both the lift $\tilde f$ chosen, and the parameter x. Proof of this fact may be found in Proposition 11.1.1 of [8] from which the definitions of lift and rotation number are taken. It is immediately clear that the rotation number has the desirable property of identifying which maps admit periodic orbits; that is, maps f for which $f^q(x)=x$ for some $x\in S^1$ and some $q\in\mathbb{N}$, and hence $\rho(f)\in\mathbb{Q}$.

Returning to the question of polygons inscribed in K, consider the action of $T_{K,B}$ restricted to ∂K by using the following map $\tilde T$ defined on the unit circle; for convenience make the obvious change of variable so that $\theta\in[0,1)$.

$$\tilde T(\theta)=\theta+(2\pi)^{-1}\tan^{-1}(l(2\pi\theta)-r(2\pi\theta)).$$

If the rotation number $\rho(\tilde T)$ is irrational, there can be no periodic points of $T_{K,B}$ on ∂K. On the other hand, suppose that $\rho(\tilde T)=p/q$ with p and q coprime. Then any periodic point x of $T_{K,B}$ on ∂K must satisfy $T^q_{K,B}(x)=x$ and $T^r_{K,B}(x)\neq x$ for $0<r<q$. The following lemma states this fact in a geometric context.

Lemma 1. *Let K be a planar convex body containing the unit ball B in its interior. Suppose that P and Q are polygons inscribed in K with edges supporting B. Then P and Q have the same number of vertices.*

Lemma 1 now permits the following definition. Let K be a planar convex body containing the unit ball in its interior; if there is a polygon P inscribed in K with edges supporting B, let $N(K)$ be the number of vertices of P; if no such polygon exists, let $N(K) = -1$.

4.4. A necessary condition

The relevance of the preceding section to Conjecture 2 (when $L = B$) can now be demonstrated. The following lemma shows that there is a class of convex bodies for which the conjecture holds subject to the restriction $L = B$. Namely, convex bodies K for which $N(K)$ is odd.

Lemma 2. *Suppose that $K \neq M \subset \mathbb{R}^2$ are convex bodies with $B \subset \operatorname{int} K \cap \operatorname{int} M$, and that $|K \cap l| = |M \cap l|$ whenever l is a line supporting B. Then $N(K)$ and $N(M)$ are equal and even.*

Proof. Use the chord-map $T = T_{K,B} = T_{M,B}$ defined as in (2) using an appropriate definition for l. Using the fact that T leaves both ∂K and ∂M invariant, it follows immediately that $N(K) = N(M)$; that $N(K)$ is even will follow easily once it is established that $N(K) > 0$. In order to do this, an appropriate polygon will be constructed. The method used was proposed by C.A. Rogers.

It is shown first that, neglecting a reversal of orientation, T preserves a measure on $\mathbb{R}^2 \backslash B$. Introduce a conjugate set of coordinates, $[\phi, s]$, given by

$$x[\phi, s] = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} + s \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}.$$

Let $D \subset \mathbb{R}^2 \backslash B$ be a region and claim that

$$\mu(D) := \int_{x[\phi,s]\in D} \mathrm{d}\phi\,\mathrm{d}s = \int_{x(\theta,r)\in D} \mathrm{d}\theta\,\mathrm{d}r. \tag{3}$$

This is easy to see, for $x[\phi, s] = x(\theta, r)$ precisely when $s = r$ and $\phi = \theta - 2\tan^{-1} r$. Hence

$$\left|\frac{\partial(\phi, s)}{\partial(\theta, r)}\right| = \begin{vmatrix} 1 & -2/(1+r^2) \\ 0 & 1 \end{vmatrix} = 1,$$

so (3) holds as claimed.

Next, observe that the boundaries of K and M coincide at some point. Using the coordinates (θ, r) parameterise ∂K and ∂M by $r(\theta)$ and $\tilde{r}(\theta)$, respectively, and assume without loss of generality that $r(0) = \tilde{r}(0)$. Further assume that for small positive excursions of θ, say $\delta > \theta > 0$, the boundaries satisfy $\tilde{r}(\theta) > r(\theta)$. Consider the region D, defined by

$$D = \{x(\theta, r) \mid \theta \in [0, \delta] \text{ and } r \in [r(\theta), \tilde{r}(\theta)]\}.$$

Clearly, using the definitions of the coordinate systems (θ, r) and $[\phi, s]$,

$$T(D) = \{x[\phi, s] \mid \phi \in [0, \delta] \text{ and } s \in [l(\phi) - \tilde{r}(\phi), l(\phi) - r(\phi)]\}.$$

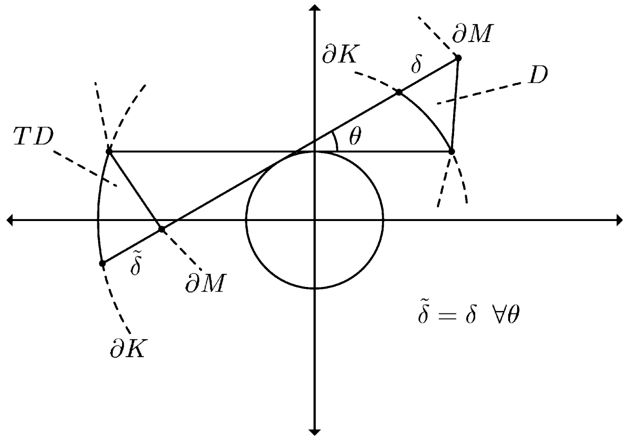

Fig. 3. The measure preserving nature of T.

See Fig. 3 for an illustration of the method. Hence, using (3),

$$\mu(T(D)) = \int_{T(D)} \mathrm{d}\theta\, \mathrm{d}r = \int_{T(D)} \mathrm{d}\phi\, \mathrm{d}s = \int_{D} \mathrm{d}\theta\, \mathrm{d}r = \mu(D),$$

showing that the regions D and $T(D)$ have the same measure with respect to μ. Now let

$$\theta_0 = \min\{\theta > 0 \mid \tilde{r}(\theta) = r(\theta)\},$$

$$D = \{x(\theta, r) \mid \theta \in [0, \theta_0] \text{ and } r \in [r(\theta), \tilde{r}(\theta)]\}.$$

By hypothesis, $\theta_0 > 0$ and hence $\mu(D) > 0$. Consider the iterates of D under T. If $T^k(D) \cap T^m(D)$ has empty interior for all k and m, then putting

$$R = \mathrm{cl}(K \backslash M \cup M \backslash K),$$

it follows that $\mu(R)$ is unbounded, a clear impossibility. Therefore, without loss of generality for some k, $I = \mathrm{int}(T^k(D) \cap D)$ is non-empty. Since T is continuous and leaves ∂K invariant, at least one of $T^k(0, r(0))$ or $T^k(\theta_0, r(\theta_0))$ lies on ∂K between $(0, r(0))$ and $(\theta_0, r(\theta_0))$. However, T is orientation preserving on ∂K and by hypothesis, there is no solution of $r(\theta) = \tilde{r}(\theta)$ for $0 < \theta < \theta_0$. Hence $(0, r(0))$ is a fixed point of T^k as required. In fact, since $(\theta_0, r(\theta_0))$ lies on ∂K, it is also a fixed point of T^k and by continuity, D is invariant under T^k.

The fact that $k = N(K) = N(M)$ must be even can be seen as follows. Let x be a point of $\mathrm{int}\, D$. Then $x \in M$, but $x \notin K$. Hence, $T(x) \in K$, but $T(x) \notin M$, and so on. However D is invariant under T^k, so $T^k(x) \in M$ and k must be even.

This result suggests that the number of bodies for which Conjecture 2 *fails* must be relatively small; unless one can inscribe an even polygon in K with edges supporting B, K is determined by the lengths of its chords supporting B. One might conjecture, for example, that the set of convex bodies K for which $N(K)$ is even is of first category in the set of all convex bodies (since $N(K)$ non-negative corresponds to a rational rotation number). However, the rotation number $\rho(T)$ of T does not in general depend smoothly on T, especially at rational values of $\rho(T)$! See the discussion in Chapter 11 of [8] for details of this.

4.5. Chords meeting an annulus

The next result illustrates a modification of the problem which renders the solution trivial.

Theorem 2. *Let $K_1, K_2 \subset \mathbb{R}^2$ be a convex bodies containing B in their interiors. Suppose that whenever l is a line supporting one of the balls $(1-\lambda)B$ with $0 \leqslant \lambda \leqslant \varepsilon$, the chord-lengths $|K_1 \cap l|$ and $|K_2 \cap l|$ are equal. Then $K_1 = K_2$.*

Proof. Let $x \in \partial K_1 \cap \partial K_2 \neq \emptyset$, and for $\lambda \in [0, \varepsilon]$, let T_λ denote the chord-map $T_{K_i,(1-\lambda)B}$. It should be clear that each T_λ is continuous and invertible (at least close to the boundaries of K_1 and K_2) and thus that the set

$$I(x) = \{T_\lambda^{-1}(T_{\varepsilon/2}x) \mid \lambda \in [0, \varepsilon]\}$$

is a non-degenerate arc in $\partial K_1 \cap \partial K_2$. Furthermore, x is contained in the interior of this line segment. Hence $\partial K_1 \cap \partial K_2$ is both open and closed in ∂K_i, so $K_1 = K_2$ as required.

4.6. A ball on the boundary

The final two-dimensional result involves another restriction. Assume that the ball B touches the boundary of K. In this case K is indeed determined by its chords supporting B.

Theorem 3. *Suppose that $K_1, K_2 \subset \mathbb{R}^2$ contain the unit ball, B, and that $\partial K_1 \cap B$ and $\partial K_2 \cap B$ are single points. If, further, the lengths of the chords $K_i \cap l$ are equal whenever l is a line supporting B, then $K_1 = K_2$.*

Proof. The first step is to prove that K_1 and K_2 meet B at the same point. Suppose without loss of generality that B meets ∂K_1 at $(1,0)^{\mathrm{T}}$. Let $l(\theta)$ denote the length of the chord cutting K_1 or K_2 supporting B at $\xi(\theta) = (\cos\theta, \sin\theta)$. If $l(0) = 0$ then K_2 also meets B at $(1,0)^{\mathrm{T}}$. If $l(0) \neq 0$ then using convexity, it is easy to see that l must be discontinuous at 0. Considering K_2, this can only happen where ∂K_2 meets B, so again K_2 meets B at $(1,0)^{\mathrm{T}}$.

Let R be the region

$$R = \mathrm{cl}(K_1 \backslash K_2 \cup K_2 \backslash K_1)$$

and using the notation of Theorem 1, let

$$l^+(\theta) = \{x(\theta, r) \mid r \geqslant 0\},$$
$$l^-(\theta) = \{x(\theta, r) \mid r \leqslant 0\}.$$

Since the lengths of the chords of K_1 and K_2 supporting B match, it follows that for all θ

$$r^+(\theta) = |R \cap l^+(\theta)| = |R \cap l^-(\theta)| = r^-(\theta).$$

To prove the result, expressions for r and r^- are derived and using these it will be shown that R has empty interior.

Let C be the characteristic function defined by

$$C(\theta, v) = \begin{cases} 1 & \text{if } \sec v\xi(\theta) \in R, \\ 0 & \text{otherwise} \end{cases}$$

and note that if $v = \tan^{-1} r$

$$x(\theta, r) = \sec v\xi(\theta - v).$$

Thus, is it possible to write

$$r^+(\theta) = \int_0^{\pi/2} C(\theta - v, v) \sec^2 v \, \mathrm{d}v$$

and if $f : \mathbb{R} \to \mathbb{R}$ has period 2π

$$\begin{aligned} \langle r^+, f \rangle &:= \int_0^{2\pi} f(\theta) r^+(\theta) \, \mathrm{d}\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} C(\theta - v, v) f(\theta) \sec^2 v \, \mathrm{d}\theta \, \mathrm{d}v \\ &= \int_0^{2\pi} \int_0^{\pi/2} f(\theta + v) C(\theta, v) \sec^2 v \, \mathrm{d}\theta \, \mathrm{d}v. \end{aligned} \tag{4}$$

It may similarly be shown that

$$\langle r^-, f \rangle = \int_0^{2\pi} \int_0^{\pi/2} f(\theta - v) C(\theta, v) \sec^2 v \, \mathrm{d}\theta \, \mathrm{d}v. \tag{5}$$

Finally, by (4), (5) and the fact that $r^+ = r^-$,

$$\begin{aligned} 0 &= \langle r^+ - r^-, f \rangle \\ &= \int_0^{2\pi} \int_0^{\pi/2} (f(\theta + v) - f(\theta - v)) C(\theta, v) \sec^2 v \, \mathrm{d}\theta \, \mathrm{d}v. \end{aligned} \tag{6}$$

Next, claim that whenever $\sec v\xi(\phi) \in R$, both $\phi + v$ and $\phi - v$ lie in the range $[0, 2\pi)$. If so, set $f(\theta) = \theta$, for $\theta \in [0, 2\pi)$, and note that whenever $C(\theta, v) \neq 0$, f satisfies $f(\theta \pm v) = \theta \pm v$; in this case

$$2 \int_R v \sec^2 v \, \mathrm{d}\phi \, \mathrm{d}v = 0,$$

implying that R has empty interior and completing the proof. So it remains only to prove the claim.

That $\theta \pm v$ lie in $[0, 2\pi)$ follows from the fact that K_1 and K_2 touch B at $(1,0)^{\mathrm{T}}$. If $x = \sec v\xi(\phi) \in K_i$

$$1 \geqslant \langle \sec v\xi(\phi), \xi(0)\rangle = \sec v \cos \phi,$$

since the line supporting B at $(1,0)^{\mathrm{T}}$ is necessarily a support line of K_i for $i = 1, 2$. That is, $\cos v \geqslant \cos \phi$. The result follows.

5. Higher dimensions

So far, only the planar version of Conjecture 2 has been considered. The following section eliminates a number of possible formulations in higher dimensions.

5.1. Sections of codimension greater than 1

Suppose that $d > 2$ and $K \subset \mathbb{R}^d$ is a convex body containing the Euclidean unit ball B in its interior. Fix $1 \leqslant j \leqslant d - 1$; let F_j denote the family of j-planes supporting B.

Theorem 4. *Suppose that $K, M \subset \mathbb{R}^d$ are convex bodies containing B in their interiors. Fix $1 \leqslant j \leqslant d - 2$ and further suppose that whenever $H \in F_j$ the j-volumes $|K \cap H|$ and $|M \cap H|$ agree. Then $K = M$.*

The proof is in two stages. First a topological argument is used to prove a slightly different result, then the invertibility of the Radon transform for certain functions is used to complete the proof. The former is achieved in the following lemma.

Suppose that $d > 2$ and $C, K \subset \mathbb{R}^d$ are convex bodies with $0 \in C \subset \operatorname{int} K$. If $x, y \in \partial K$ write $x \sim y$ whenever there is a line l supporting C with $x, y \in l$.

Lemma 3. *Fix $x_0 \in \partial K$. There is a neighbourhood N of x_0 in ∂K such that whenever $y_0 \in N$ there exists $z \in \partial K$ with $x_0 \sim z$ and $y_0 \sim z$.*

Proof. Let $x_0 \in \partial K$ and let

$$S(x_0) = \{x \in \partial K \mid l(x_0, x) \text{ supports } C\},$$

where $l(x_0, x)$ denotes the line passing through x and x_0. Separate x_0 from C by a hyperplane H and let $\tilde{C}$ denote the projection of C onto H from x_0. Then $\tilde{C}$ is a $(d-1)$-dimensional convex body whose boundary is also the projection of $S(x_0)$ onto H from x_0. Thus $S(x_0)$ is a topological $(d-2)$-sphere.

$S(x_0)$ separates ∂K into two sets $U(x_0), V(x_0)$ where

$$U(x_0) = \{x \in \partial K \mid l(x_0, x) \text{ meets int } C\},$$

$$V(x_0) = \{x \in \partial K \mid l(x_0, x) \text{ does not meet } C\}.$$

Notice that $S(x_0)$ is the boundary between $U(x_0)$ and $V(x_0)$. That is,

$$S(x_0) = \mathrm{cl}(U(x_0)) \cap \mathrm{cl}(V(x_0)).$$

Let $[x_0, 0]$ meet ∂C at a. Let H_1 be a support hyperplane to C at a. Then H_1 strictly separates x_0 from 0 with x_0 in the open half space $\mathrm{int}\, H_1^+$.

Let $y_0 \in \partial K \cap \mathrm{int}\, H_1^+$. Then, by considering the 2-plane through x_0, y_0 and 0 we see that $S(y_0)$ meets both $\mathrm{cl}(U(x_0))$ and $\mathrm{cl}(V(x_0))$. Since $S(y_0)$ is a connected set and $S(x_0)$ separates $\mathrm{cl}(U(x_0))$ and $\mathrm{cl}(V(x_0))$ there exists $z \in S(y_0) \cap S(x_0)$. Hence if $y_0 \in \partial K \cap H_1^+$, there exists $z \in \partial K$ such that $x_0 \sim z$ and $z \sim y_0$ as required.

Before proving Theorem 4 it is necessary to mention one further result. Suppose that $K \subset \mathbb{R}^d$ is a convex body with $0 \in \mathrm{int}\, K$. If j is an integer, and $\rho_K : S^{d-1} \to \mathbb{R}$ is the radial function of K, define

$$l_K^j(u) = \rho_K^j(u) + \rho_K^j(-u).$$

This function is often referred to as the j-chord function. Larman and Tamvakis [11] proved that l_K^{d-1} can be constructed using knowledge only of $f_{K,\{0\}}$. That is, if the volumes of the one-codimensional sections of K which contain 0 are given, the $(d-1)$-chord function can be reconstructed. Several other authors give similar results. Gardner [5] (Theorem 7.2.3) provides a more general result, and of particular interest in subsequent sections, Schneider [13] gives a method using spherical harmonics.

The preparations for the proof of Theorem 4 are now complete.

Proof. First consider the case $j = 1$. In this case, whenever l is a line supporting B, the chords $K \cap l$ and $M \cap l$ have the same length. Suppose $x \in \partial K \cap \partial M$, and l is a line supporting B with $x \in l$. Then the other endpoint of the segment $K \cap l$ also belongs to ∂M. Hence, using Lemma 3, with C replaced by B, we see that $\partial K \cap \partial M$ is open in ∂K. By definition it is also true that $\partial K \cap \partial M$ is closed in ∂K. Hence $\partial K \cap \partial M$ is either empty or is equal to ∂K. However, the boundaries of K and M must meet. Hence

$$\partial K \cap \partial M = \partial K = \partial M,$$

which completes the proof.

Next, consider the other cases; fix $1 < j < d - 1$. Suppose that l is a line supporting B at u and meeting ∂K at x and y. Set

$$g_K(l) = \|x - u\|^j + \|y - u\|^j$$

and define g_M similarly. It is claimed that for all l supporting B the identity $g_K(l) = g_M(l)$ holds. If so, an argument similar to that for the case $j = 1$ completes the proof. $\square$

It remains, therefore, to prove the claim. Identify H with $\mathbb{R}^{j+1}$ with the origin at u. Clearly, $H \cap K$ and $H \cap M$ are convex bodies in $\mathbb{R}^{j+1}$ containing 0 in their interiors.

By hypothesis, if $F \subset \mathbb{R}^{j+1}$ is a 1-codimensional subspace,

$$|K \cap F| = |M \cap F|.$$

Thus, using [5, Theorem 7.2.3] for example, if $l \subset H$ is a line supporting B at u then $g_K(l) = g_M(l)$ as required. The choice of H was arbitrary so the result follows.

5.2. Sections of codimension one

In this section, the following question is considered. Suppose that K is a convex body containing the unit ball in its interior. To what extent is K determined by the volumes of its sections supporting the unit ball?

As in the two-dimensional case, it has not been possible to provide a complete answer to this question. In fact, the best result at present is the following analogue of Theorem 2. If $K \subset \mathbb{R}^d$ is a convex body, and $x \neq 0$, let $K(x)$ denote the one-codimensional section of K cut by the hyperplane perpendicular to and containing x.

Theorem 5. *Suppose that $K, M \subset \mathbb{R}^d$ are convex bodies with $B \subset \operatorname{int} K \cap \operatorname{int} M$ and that d is odd. Let $A \subset (0,1]$ be (at least) countably infinite and suppose that whenever $\mu \in A$ and $u \in S^{d-1}$ the volumes $|K(\mu u)|$ and $|M(\mu u)|$ are equal. Then $K = M$.*

Before proving the result, it is appropriate to make two remarks concerning its scope. First, we do not believe that the condition on d is significant. It would certainly be interesting to find counterexamples when d is even. However, it will become clear that the requirement on d is used merely to render the calculations tractable. Second, it is likely that this result could be stated for suitably constrained star bodies. However, it should be noted that the proof of Theorem 5 would not accommodate an extension to cover more general distributions.

Proof. In order to prove the result, use will be made of spherical harmonics and in particular results in [13]. It will be shown that with the given information about K it is possible to reconstruct the radial function of K.

To begin with, let $f(x) = |K(x)|$. The first task is to derive an expression for

$$F(s_n, \mu) = \int_{S^{d-1}} f(\mu u) s_n(u) \, \mathrm{d}\omega_{d-1}(u),$$

where s_n is a spherical harmonic of degree n. In order to do this write

$$f(\mu u) = \lim_{h \to 0} \frac{1}{2h} \int_K \delta_h(x, u, \mu) \, \mathrm{d}\lambda_d(x), \tag{7}$$

where δ_h is the function defined by

$$\delta_h(x, u, \mu) = \begin{cases} 1, & \langle x, u \rangle \in [\mu - h, \mu + h], \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that since $B \subset \operatorname{int} K$, the convergence to the limit in (7) is uniform over S^{d-1} when $\mu \in [0,1]$. Hence

$$F(s_n,\mu) = \int_{S^{d-1}} s_n(u) \left(\lim_{h\to 0} \frac{1}{2h} \int_K \delta_h(x,u,\mu)\,\mathrm{d}\lambda_d(x) \right) \mathrm{d}\omega_{d-1}(u)$$

$$= \int_K \left(\lim_{h\to 0} \frac{1}{2h} \int_{S^{d-1}} \delta_h(x,u,\mu) s_n(u)\,\mathrm{d}\omega_{d-1}(u) \right) \mathrm{d}\lambda_d(x). \tag{8}$$

The next task is to evaluate the inner integral and limit as $h \to 0$. Write

$$x = t^{-1} u_0$$

with $u_0 \in S^{d-1}$ and $t > 0$. Note that if $t > \mu^{-1}$ then for sufficiently small $h > 0$, δ_h is zero. Assume in what follows that $t < \mu^{-1}$. For $h > 0$ define

$$I(h) = \int_{u\in S^{d-1}} \delta_h(t^{-1}u_0, u, \mu) s_n(u)\,\mathrm{d}\omega_{d-1}(u).$$

To partially evaluate $I(h)$ write the variable of integration u in the form

$$u = \cos\theta u_0 + \sin\theta v,$$

with $\theta \in [0,\pi]$ and $v \in S^{d-2}(u_0)$, where

$$S^{d-2}(w) := \{ v \in S^{d-1} \mid \langle v, w\rangle = 0 \}.$$

Notice that using this parameterisation,

$$\delta_h(x,u,\mu) = \begin{cases} 1, & \cos\theta \in [t(\mu-h), t(\mu+h)], \\ 0 & \text{otherwise,} \end{cases}$$

so $I(h)$ can now be written in the form

$$I(h) = \int_{\cos^{-1}(t(\mu+h))}^{\cos^{-1}(t(\mu-h))} \sin^{d-2}\theta \int_{S^{d-2}(u_0)} s_n(\cos\theta u_0 + \sin\theta v)\,\mathrm{d}\omega_{d-2}(v)\,\mathrm{d}\theta.$$

Now appealing to the results derived in [13], the inner integral in this last expression can be rewritten as

$$\int_{S^{d-2}(u_0)} s_n(\cos\theta u_0 + \sin\theta v)\,\mathrm{d}\omega_{d-2}(v) = |\omega_{d-2}| C_n^{\nu}(1)^{-1} C_n^{\nu}(\cos\theta) s_n(u_0), \tag{9}$$

where $|\omega_{d-2}|$ is the surface area of the unit sphere in $\mathbb{R}^{d-1}$, $\nu = (d-2)/2$ and C_n^{ν} is the nth Gegenbauer polynomial of order ν. Hence

$$I(h) = \gamma_n s_n(u_0) \int_{\cos^{-1}(t(\mu+h))}^{\cos^{-1}(t(\mu-h))} \sin^{d-2}\theta C_n^{\nu}(\cos\theta)\,\mathrm{d}\theta$$

with $\gamma_n = |\omega_{d-2}| C_n^{\nu}(1)^{-1}$. Now it is possible to evaluate, using de l'Hospital's rule, the limit

$$\lim_{h\to 0} \frac{1}{2h} I(h) = \gamma_n (1 - \mu^2 t^2)^{(d-3)/2} C_n^{\nu}(\mu t) s_n(u_0).$$

Next, recalling that $x = t^{-1}u_0$, (8) can be written as

$$F(s_n, \mu) = \gamma_n \int_{S^{d-1}} s_n(u) \int_{\rho_K(u)^{-1}}^{\mu^{-1}} t^{-(d+1)}(1-\mu^2t^2)^{(d-3)/2} C_n^{\nu}(\mu t)\,\mathrm{d}t\,\mathrm{d}\omega_{d-1}(u). \quad (10)$$

This equation shows the complexity of the relationship between K and the spherical harmonics of the function $f_{K,B}$. It is apparent that spherical harmonics are, perhaps, not the most efficient tool for deducing the properties of K from $f_{K,B}$. However it is possible, using our strong hypotheses, to begin to unravel this relation.

The key observation is that when $d \geqslant 3$ is odd, the exponent $(d-3)/2$ is an integer. Hence the integrand in (10) may be written as the product of $t^{-(d+1)}s_n(u)$ and a homogeneous polynomial

$$(1-\mu^2t^2)^{(d-3)/2} C_n^{\nu}(\mu t) = \sum_{k=0}^{n+d-3} \lambda_n^k \mu^k t^k. \quad (11)$$

It is not necessary to calculate all the values of λ_n^k above; it will suffice to show that certain of them are non-zero. For the moment, it is enough to note that now $F(s_n, \mu)$ can be written in the following form:

$$\begin{aligned} F(s_n, \mu) &= \gamma_n \sum_{k=0}^{n+d-3} \lambda_n^k \mu^k \int_{S^{d-1}} s_n(u) \int_{\rho_K(u)^{-1}}^{\mu^{-1}} t^{k-d-1}\,\mathrm{d}t\,\mathrm{d}\omega_{d-1}(u) \\ &= \gamma_n \sum_{k=0}^{n+d-3} \lambda_n^k \mu^k (B_k(s_n, \mu) - A_k(s_n)), \end{aligned} \quad (12)$$

where

$$A_k(s_n) = \begin{cases} \dfrac{1}{k-d} \displaystyle\int_{S^{d-1}} \rho_K^{d-k}(u) s_n(u)\,\mathrm{d}\omega_{d-1}(u), & k \neq d, \\ \displaystyle\int_{S^{d-1}} \log(\rho_K(u)) s_n(u)\,\mathrm{d}\omega_{d-1}(u), & k = d \end{cases}$$

and

$$B_k(s_n, \mu) = \begin{cases} \dfrac{1}{k-d} \displaystyle\int_{S^{d-1}} \mu^{d-k} s_n(u)\,\mathrm{d}\omega_{d-1}(u), & k \neq d, \\ \displaystyle\int_{S^{d-1}} \log(\mu) s_n(u)\,\mathrm{d}\omega_{d-1}(u), & k = d. \end{cases}$$

Notice that $B_k(s_n, \mu)$ is independent of K. In fact it is zero unless $n = 0$. Hence B_k may be treated as known, and it is possible to determine the values of the function G defined by

$$G(s_n, \mu) = \sum_{k=0}^{n+d-3} \lambda_n^k \mu^k B_k(s_n, \mu) - F(s_n, \mu),$$

for all spherical harmonics s_n, and countably many values of $\mu \in [0, 1]$. From (12), however,

$$G(s_n, \mu) = \gamma_n \sum_{k=0}^{n+d-3} \lambda_n^k \mu^k A_k(s_n).$$

That is, G is polynomial in μ. Therefore, since $G(s_n, \mu)$ is known for countably many values of μ, it is possible to determine $G(s_n, \cdot)$ completely; or equivalently the hypotheses make it possible to construct the coefficients of G given by

$$\alpha_k(s_n) = \lambda_n^k A_k(s_n).$$

It will soon become clear that this is enough to reconstruct K. First claim that for n even, λ_n^0 is non-zero, and that for n odd λ_n^1 is non-zero. If so given any spherical harmonics s_{2n} and s_{2n+1} of degrees $2n$ and $2n+1$, respectively, it is possible to construct the scalar products

$$\langle \rho_K^d, s_{2n} \rangle = \int_{S^{d-1}} \rho_K^d(u) s_{2n}(u) \, \mathrm{d}\omega_{d-1}(u),$$
$$\langle \rho_K^{d-1}, s_{2n+1} \rangle = \int_{S^{d-1}} \rho_K^{d-1}(u) s_{2n+1}(u) \, \mathrm{d}\omega_{d-1}(u).$$

Using the completeness of the odd and even systems of spherical harmonics and the uniform continuity of ρ_K, the functions

$$\begin{aligned} \phi(u) &= \rho_K^d(u) + \rho_K^d(-u), \\ \tilde{\phi}(u) &= \rho_K^{d-1}(u) - \rho_K^{d-1}(-u) \end{aligned} \tag{13}$$

are determined for all $u \in S^{d-1}$. It is now an easy matter to determine ρ_K, for if

$$\alpha^d + \beta^d = a,$$
$$\alpha^{d-1} - \beta^{d-1} = b,$$

with $\alpha, \beta > 0$, then

$$\beta = (\alpha^{d-1} - b)^{1/(d-1)}.$$

Therefore

$$\alpha^d + (\alpha^{d-1} - b)^{d/(d-1)} = a$$

and differentiation with respect to α of this last expression gives

$$(d-1)\alpha^{d-1} + (\alpha^{d-1} - b)^{1/(d-1)} d\alpha^{d-2} > 0.$$

Hence the solution is unique, and ρ_K is determined as claimed.

To complete the proof it suffices to verify the claim that λ_{2n}^0 and λ_{2n+1}^1 are non-zero. In order to see this, recall the definition of λ_n^k as coefficients of the polynomial in (11).

$$\lambda_{2n}^0 = C_{2n}^v(0).$$

The Gegenbauer polynomials C_n^v may be defined in terms of the expansion

$$\frac{1}{(1 - 2tx + x^2)^v} = \sum_{n=0}^{\infty} C_n^v(t) x^k,$$

from which it follows that $C_{2n}^{v}(0)$ is non-zero. Also

$$\begin{aligned}\lambda_{2n+1}^{1} &= \frac{\mathrm{d}}{\mathrm{d}z}((1-z^2)^{(d-3)/2}C_{2n+1}^{v}(z))\Big|_{z=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}z}C_{2n+1}^{v}(z)\Big|_{z=0}\\ &= vC_{2n}^{v+1}(0)\end{aligned}$$

and this is also non-zero since $v=(d-2)/2$ and d is odd. This completes the result. □

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